# Generalised definitions of certain functions and their uses

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**Abstract**

Generalised definitions of exponential, trigonometric sine and cosine and hyperbolic sine and cosine functions are given. In the lowest order, these functions correspond to ordinary exponential, trigonometric sine etc. Some of the properties of the generalised functions are discussed. Importance of these functions and their possible applications are also considered.

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# 1 Introduction

Ordinary differential equations (ODEs) are ubiquitous in various branches of physical and biological sciences, and are essential to our understanding of topics as diverse as stability theory, analysis of electrical signals and networks, chaos and nonlinear dynamics, radiative processes, etc.[1, 2, 3, 4, 5, 6, 7, 8, 9]. In this paper we try to generalize the definition of *sine*, *cosine* and *hyperbolic* functions as solutions of differential equation of the form $y_{xx} + f(x)y = 0$. For $f(x) \propto x, x^{-1}, x^2, x^{-2}$ the equations can be brought to Airy type, Coulomb type, Harmonic oscillator type and Euler type equations having wide application in physical science problems. We show that many well known ODE solutions of physical science can also be defined in terms of these generalized functions which we call now as $g$ functions.

The organization of the paper is as follows. In section 2 we formulate the definition of $g$ functions related to generalised exponential, generalised cosine and sine and generalised cosine hyperbolic and sine hyperbolic functions. In section 3 we discuss some important ODEs of mathematical physics in terms of $g$ functions. In section 4 we end up with a concluding discussion.

# 2 Formulation of $g$ function

We begin with formal definitions of exponential, trigonometric and hyperbolic functions as follows. The exponential function may be defined as solution of the differential equation

$$\frac{dy}{dx} = y \tag{1}$$

Similarly, trigonometric sine and cosine functions may be defined as solutions to the following differential equation

$$\frac{d^2y}{dx^2} = -y \tag{2}$$

while hyperbolic sine and cosine functions are defined as solutions of

$$\frac{d^2y}{dx^2} = y \tag{3}$$

Let us consider solutions to the following set of differential equations

$$\begin{aligned} \frac{dy}{dx} &= x^n y && (4) \\ \frac{d^2y}{dx^2} &= -x^n y && (5) \\ \frac{d^2y}{dx^2} &= x^n y && (6) \end{aligned}$$

**Case I : Generalised exponential function**

Following Forsyth[5], we express gerenal solutions of (4) as

where S denotes the operation of integration from 0 to x i.e. $\int_0^x dx$ and $P = x^n$. Then the solution of (4) becomes

$$y = 1 + \frac{x^{(n+1)}}{(n+1)} + \frac{x^{(2n+2)}}{(n+1)(2n+2)} + \frac{x^{(3n+3)}}{(n+1)(2n+2)(3n+3)} + ... \quad (8)$$

The series on RHS of (8) is convergent for finite x and n. The only restriction on n is that $n \neq -1$. We can also solve equation (4) by separation of variables. The result is

$$lny = \frac{x^{(n+1)}}{(n+1)}$$

so that

$$y = e^{\frac{x^{(n+1)}}{n+1}} \quad (9)$$

It is easily verified that the RHS of the two expressions, (8) and (9), are identical for finite x and n. These can be used to define exponential function of order n as

$$e_n(x) = 1 + \frac{x^{(n+1)}}{(n+1)} + \frac{x^{(2n+2)}}{(n+1)(2n+2)} + \frac{x^{(3n+3)}}{(n+1)(2n+2)(3n+3)} + ... \quad (10)$$

Note that the ordinary exponential function is zeroth order of this generalised exponential function $e_n(x)$. Indeed, with n = 0, the above expression becomes

$$e_0(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ....$$

However, the generalised exponential allows expressions for other values of n. Behaviour of $e_n(x)$ for some values of n are shown in Figure 1.

**Case II : Generalised cosine and sine function**

Let us solve equation (5) with the same technique. We get two independent solutions as

$y_1 = 1 + S^2P + S^2PS^2P + S^2PS^2PS^2P + ....$
and
$y_2 = x + S^2Px + S^2PS^2Px + S^2PS^2PS^2Px + ....$

Here S = $\int_0^x dx$ as before and P = $-x^n$. Then these solutions define generalised cosine and sine functions for us as

$$g_c^n(x) = 1 - \frac{x^{(n+2)}}{(n+1)(n+2)} + \frac{x^{(2n+4)}}{(n+1)(n+2)(2n+3)(2n+4)} - +... \quad (11)$$

$$g_s^n(x) = x - \frac{x^{(n+3)}}{(n+2)(n+3)} + \frac{x^{(2n+5)}}{(n+2)(n+3)(2n+4)(2n+5)} - +... \quad (12)$$

these functions are identical with ordinary trigonometric cosine and sine functions respectively. But expressions for other values of n are allowed. Behaviour of these functions for some values of n are shown in Figures 2 and 3 respectively.

We now extend the above definitions for $g_c^n(x)$ and $g_s^n(x)$ for negative values of n as follows. $g_c^{-m}(x) = x g_s^{m-4}(\frac{1}{x})$ leads to

$$g_c^{-n}(x) = 1 - \frac{x^{(2-n)}}{(n-1)(n-2)} + \frac{x^{(4-2n)}}{(n-1)(n-2)(2n-3)(2n-4)} - +... \quad (13)$$

$$g_s^{-n}(x) = x - \frac{x^{(3-n)}}{(n-2)(n-3)} + \frac{x^{(5-2n)}}{(n-2)(n-3)(2n-4)(2n-5)} - +... \quad (14)$$

Note that these expressions of $g^{-n}$ are also trivially obtained by substituting -n for n in corresponding expressions for $g^n$. It must be noted that $g_{s,c}^n$ are not defined for all negative values of n, while $g_c^{-n}(x)$ is not defined for n =1 and 2, $g_s^{-n}(x)$ is not defined for n = 2 and 3. These functions are plotted for some negative n values in Figures 4 and 5.

**Case III: Generalised hyperbolic function**

For $P = x^n$, all the signs in the two series above becomes positive, and we get as solutions of equation (6),

$$g_{hc}^n(x) = 1 + \frac{x^{(n+2)}}{(n+1)(n+2)} + \frac{x^{(2n+4)}}{(n+1)(n+2)(2n+3)(2n+4)} + ... \quad (15)$$

$$g_{hs}^n(x) = x + \frac{x^{(n+3)}}{(n+2)(n+3)} + \frac{x^{(2n+5)}}{(n+2)(n+3)(2n+4)(2n+5)} + ... \quad (16)$$

where we have identified these solutions as generalised hyperbolic cosine and sine functions respectively. Note that these functions are also not defined for $n \leq 0$ integers. It is trivial to show that $g_{hc}^0(x) = cosh(x)$ and $g_{hs}^0(x) = sinh(x)$. $g_{hc}^n(x)$ is plotted for some values of n in Figure 6. The behaviour of $g_{hs}^n(x)$, for the same n values, is very similar and is therefore not shown separately.

We extend these definitions for generalised hyperbolic functions to negative n values in the same manner as was done for generalised trigonometric functions. In other words

$$g_{hc}^{-n}(x) = 1 + \frac{x^{(2-n)}}{(n-1)(n-2)} + \frac{x^{(4-2n)}}{(n-1)(n-2)(2n-3)(2n-4)} + ... \quad (17)$$

$$g_{hs}^{-n}(x) = x + \frac{x^{(3-n)}}{(n-2)(n-3)} + \frac{x^{(5-2n)}}{(n-2)(n-3)(2n-4)(2n-5)} + ... \quad (18)$$

define the generalised hyperbolic functions for negative values of n. As before,

case of $g_c^{-n}$ and $g_s^{-n}$ apply for these functions as well i.e. $n \neq 1, 2$ for $g_{hc}^{-n}(x)$ and $n \neq 2, 3$ for $g_{hs}^{-n}(x)$. Behaviour of these functions for some negative n values are shown in Figures 7 and 8.

It is clear that the above generalised definitions hold for positive and negative real values of n (except those mentioned above). Solutions to the second order ODEs with $-x^n y$ on RHS are oscillatory. The oscillatory nature of these generalised functions is tested as follows. Leighton's oscillatory theorem[10] asserts that a differential equation

$$(p(x)y') + q(x)y = 0$$

is oscillatory in $(0, \infty)$ provided $\int_0^\infty \frac{1}{p(x)}dx \to \infty$ and $\int_0^\infty q(x)dx \to \infty$. As both these conditions are satisfied in this case, the ODEs in (5) and (6) and hence their solutions are oscillatory.

It can be proved that $g_c^n(x)$ and $g_s^n(x)$, as also $g_{hc}^n(x)$ and $g_{hs}^n(x)$ are mutually orthogonal in $(-\infty, \infty)$. This shall be dealt with in a later communication. We now propose that the functions $e_n(x)$, $g_c^n(x)$, $g_s^n(x)$, $g_{hc}^n(x)$ and $g_{hs}^n(x)$ be considered as generalised exponential, generalised cosine and sine, and generalised hyperbolic cosine and sine functions respectively. From plots of these functions in Figures 1 to 5, identity of these functions with ordinary exponential, trigonometric cosine and sine, and hyperbolic cosine and sine functions for n = 0, and departure from the latter functions for higher values of n are evident.

# 3 Applications

**(I)** We next show the utility of these functions. Consider any second order ODE of the form

$$\frac{d^2u}{dx^2} + 2R\frac{du}{dx} + (R^2 + \frac{dR}{dx} + x^n)u = 0 \tag{19}$$

Substituting $u = ye^{-\int Rdx}$, R=R(x), the above ODE becomes identical with equation (5), whose solutions (except for negative integral n) can be written in terms of $g_c^n(x)$ and $g_s^n(x)$. Therefore (except for some $n \leq o$ integers), solutions of (19) can be expressed in terms of the latter functions.

**(II)** These oscillatory functions can also serve as solutions to the following type of ODEs

$$\frac{d^2y}{dx^2} + f(x)y = 0$$

where $f(x) = Ax^m + Bx^n +$ .... where A, B, .. are constants and m, n, ... may be relatively coprime.

Mathieu's etc. We discuss briefly on these connections, details will be placed in future works, with applications of $g_c^n(x)$ and $g_s^n(x)$ wherever possible. Here, we shall show how these functions can be used as solutions of Bessel's equation. Bessel's ODE is written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

Putting $x = 2\nu r^{\frac{1}{2\nu}}$ and $y = ur^{-\frac{1}{2}}$, the above ODE becomes

$$\frac{d^2u}{dr^2} + r^{\frac{1}{\nu-2}}u = 0$$

the solution of which can be written as

$$u(r) = Ag_c^{\frac{1}{\nu-2}}(r) + Bg_s^{\frac{1}{\nu-2}}(r)$$

Hence, the general solution of Bessel's ODE can be expressed as

$$y(x) = (\frac{2\nu}{x})^{\nu}[Ag_c^{\frac{1}{\nu-2}}(\frac{x}{2\nu})^{2\nu} + Bg_s^{\frac{1}{\nu-2}}(\frac{x}{2\nu})^{2\nu}]$$

**(IV)** Let us discuss the solution of Ricatti equation in terms of $g$ function. The Ricatti equation is

$$\frac{dy}{dx} + y^2 + x^m = 0 \tag{20}$$

With $\frac{u'}{u} = y$, the above equation can be cast into the form

$$\frac{u''}{u} + x^m = 0 \tag{21}$$

which evidently has two solutions in terms of $g_s^n$ and $g_c^n$.

**(V)** The confluent hypergeometric equation is

$$x\frac{d^2y}{dx^2} + (k - x)\frac{dy}{dx} - ay = 0 \tag{22}$$

If we substitute $x = -t^2/4$, $y = t^{-n}z$ with $k = n + 1$, the confluent hypergeometric equation is reduced to Bessel differential equation

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + (t^2 - n^2)z = 0 \tag{23}$$

and can be solved in terms of $g_s^n$ and $g_c^n$ as shown earlier.

Moreover Whittaker equation and Weber-Hermite equation can also be solved in terms of $g_s^n$ and $g_c^n$. The above examples merely shows the equivalence

We now present an example where the present method is shown to be superior to existing ones. In the case of an ODE such as

$$\frac{d^2y}{dx^2} - x^{-4}y = 0$$

no regular solution can be obtained by the usual, Frobenius method. Only normal integrals exist in such cases [5]. But the present method immediately provides regular solutions in a straightforward manner. This is also apparent in cases where logarithmic terms appear in the Frobenius method which are troublesome to deal with. Such a possibility does not arise in the present method. Also, for complicated nonlinear ODEs for example, often one solution appears in published tables of integrals; the present method yields all possible solutions. We just cite one example.

Consider the equation

$$yy''' + 3y'y'' + x^m yy' = 0 \tag{24}$$

With substitution $w = yy'$, the above equation reduces to the form

$$w'' + x^m w = 0 \tag{25}$$

Obviously the solutions are $w_1 = g_s^m(x)$ and $w_2 = g_c^m(x)$.

# 4 Concluding discussion

Applicability of the generalised exponential and the generalised hyperbolic functions defined above can also be established in a similar manner. It becomes apparent that such functions can be applied to a fairly broad category of physical problems. While the present work may be considered as a brief introduction to these functions, their detailed nature and properties will be taken up in our future works. We hope to establish the usefulness of these functions, and show how solutions of various the problems of mathematical physics can be expressed in much more compact and uniform manner with their help.

We wish to emphasise that the solutions to a class of differential equations given in the present work is actually a subset of a larger class of functions. Here, we have only considered first and second order ODEs of a particular type. We wish to state that these are subsets of the following ODEs viz.

$$\frac{d^n y}{dx^n} = \pm x^m y \tag{26}$$

where n and m may both vary. We wish to express solutions to such ODEs as $g_{n,m}^{\pm,k}(x)$, where k = 0, 1, ... n. For example, $e_n(x)$ is now written as $g_{1,0}^{+,1}(x)$,

adequate for denoting solutions to lower order ODEs as shown at the beginning of this work, viz. equations (4) to (6), the need for a more general notation becomes immediately apparent as higher order ODEs are considered. For example, for

$$\frac{d^3y}{dx^3} = x^m y$$

the solutions can be written as $g_{3,m}^{+,1}(x)$, $g_{3,m}^{+,2}(x)$ and $g_{3,m}^{+,3}(x)$. Such functions will be considered in our future publications.

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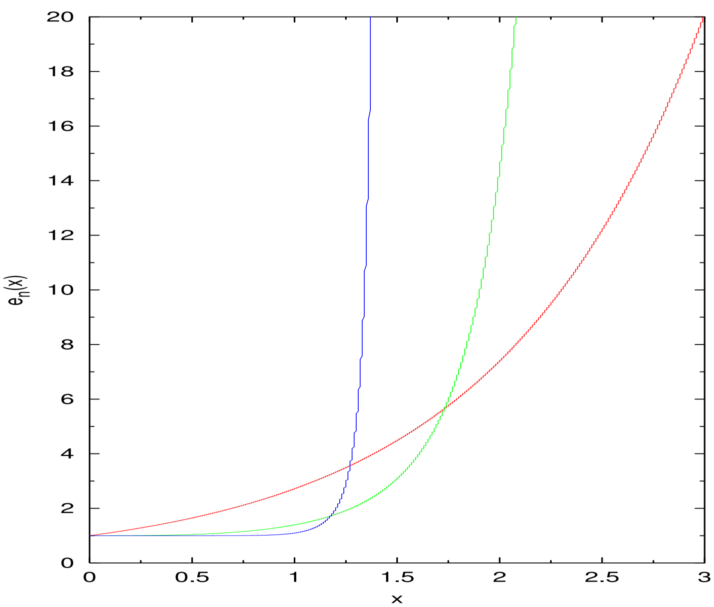


Figure 1: $e_n(x)$ vs x for n = 0 (red), 2 (green) and 10 (blue).

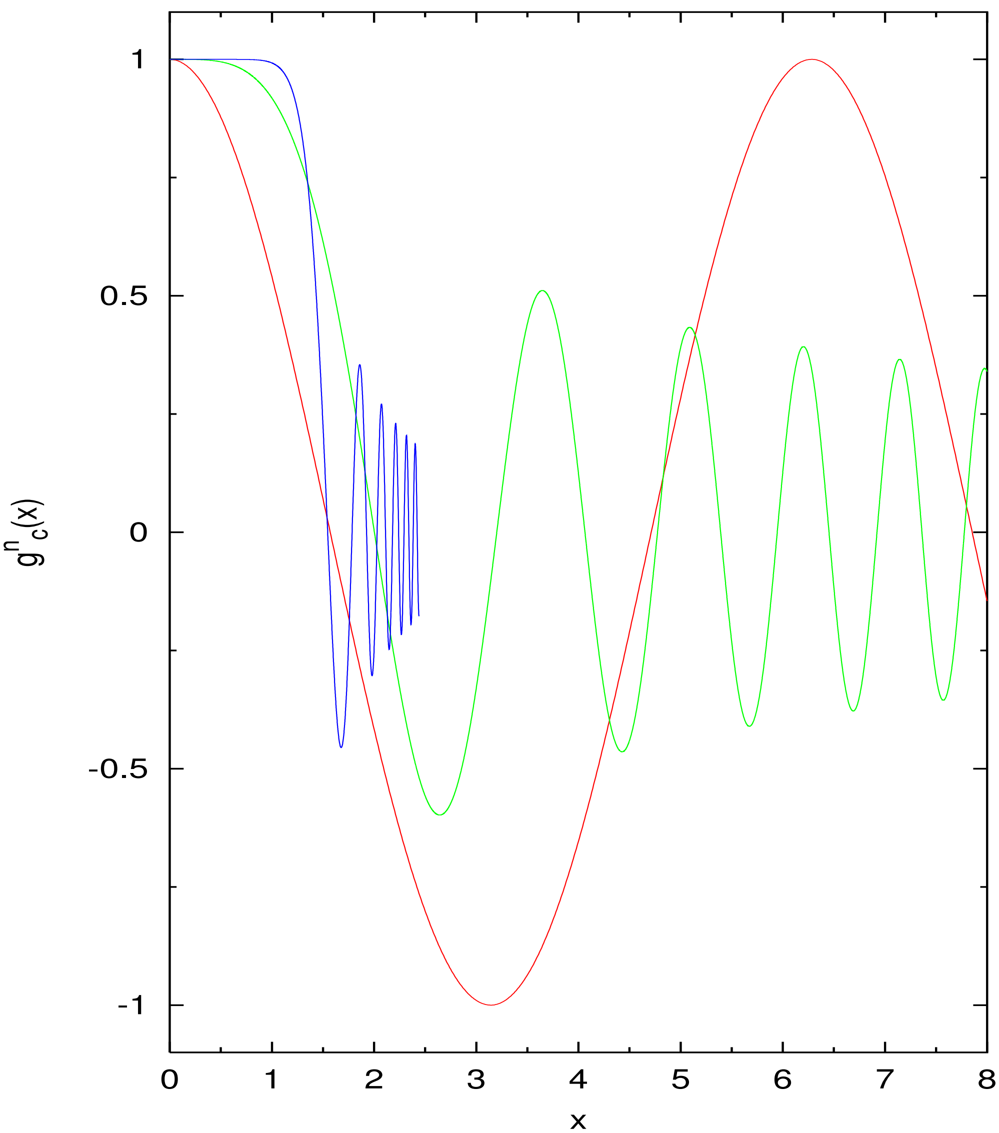


Figure 2: $g_c^n(x)$ vs x for n = 0 (red), 2 (green) and 10 (blue). Note increased oscillation as n is increased. The abrupt termination for the blue curve is due to numerical instability.

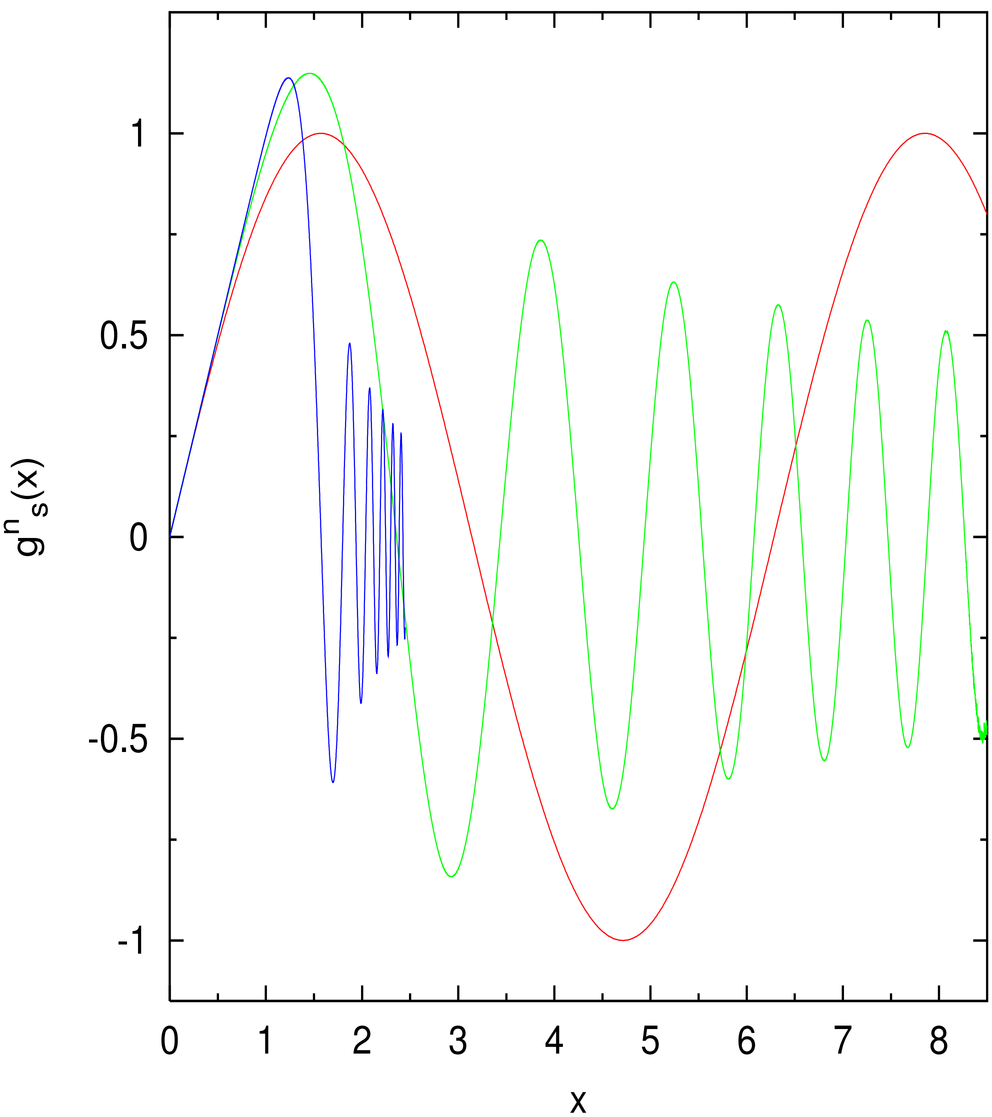


Figure 3: $g_s^n(x)$ vs x for n = 0 (red), 2 (green) and 10 (blue). Note increased oscillation as n is increased. The abrupt termination for the blue curve is due to numerical instability.

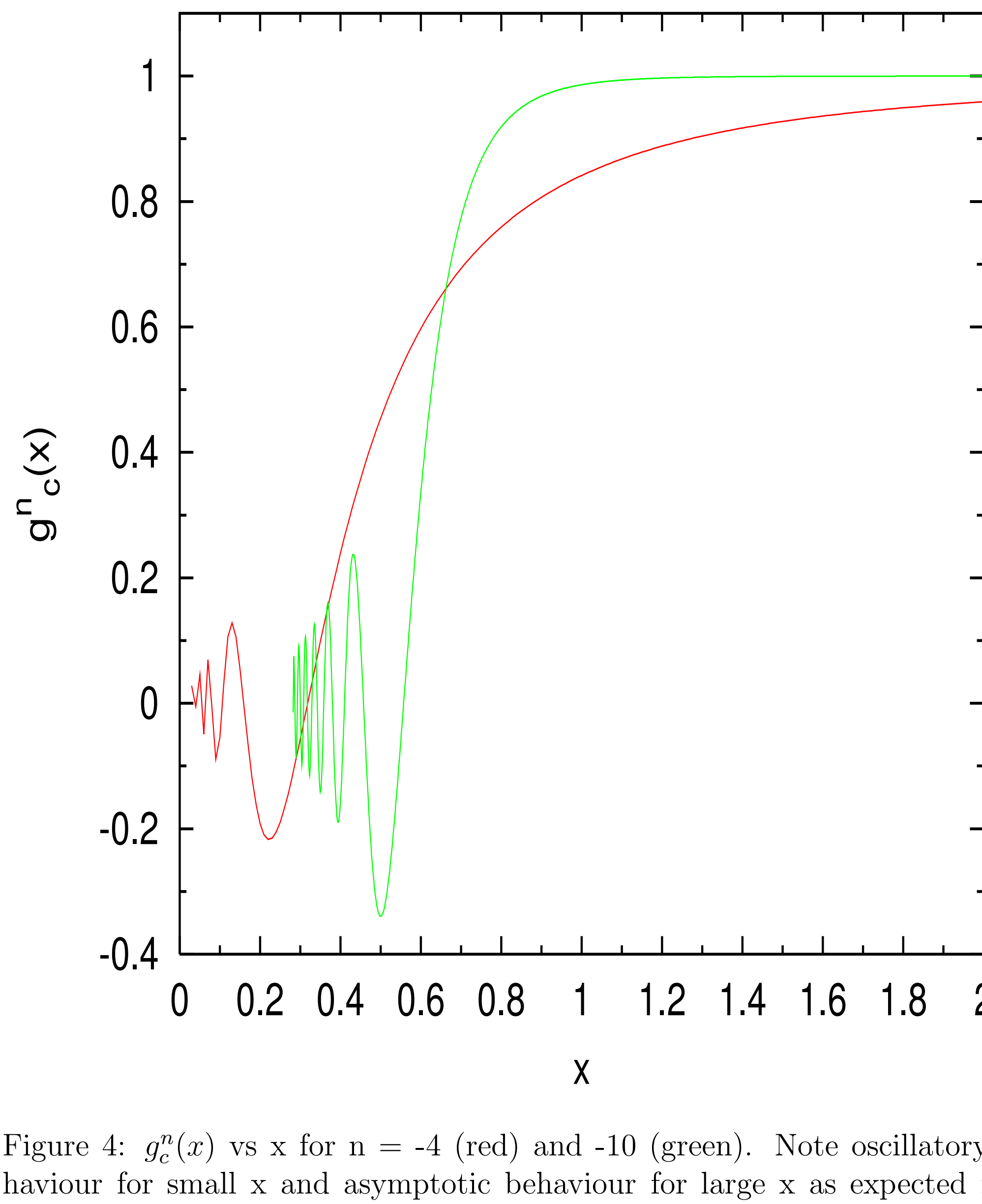


Figure 4: $g_c^n(x)$ vs x for n = -4 (red) and -10 (green). Note oscillatory behaviour for small x and asymptotic behaviour for large x as expected from equation (13). Oscillations increases as magnitude of n is increased. The abrupt termination, more apparent for n = -10 is due to numerical instability.

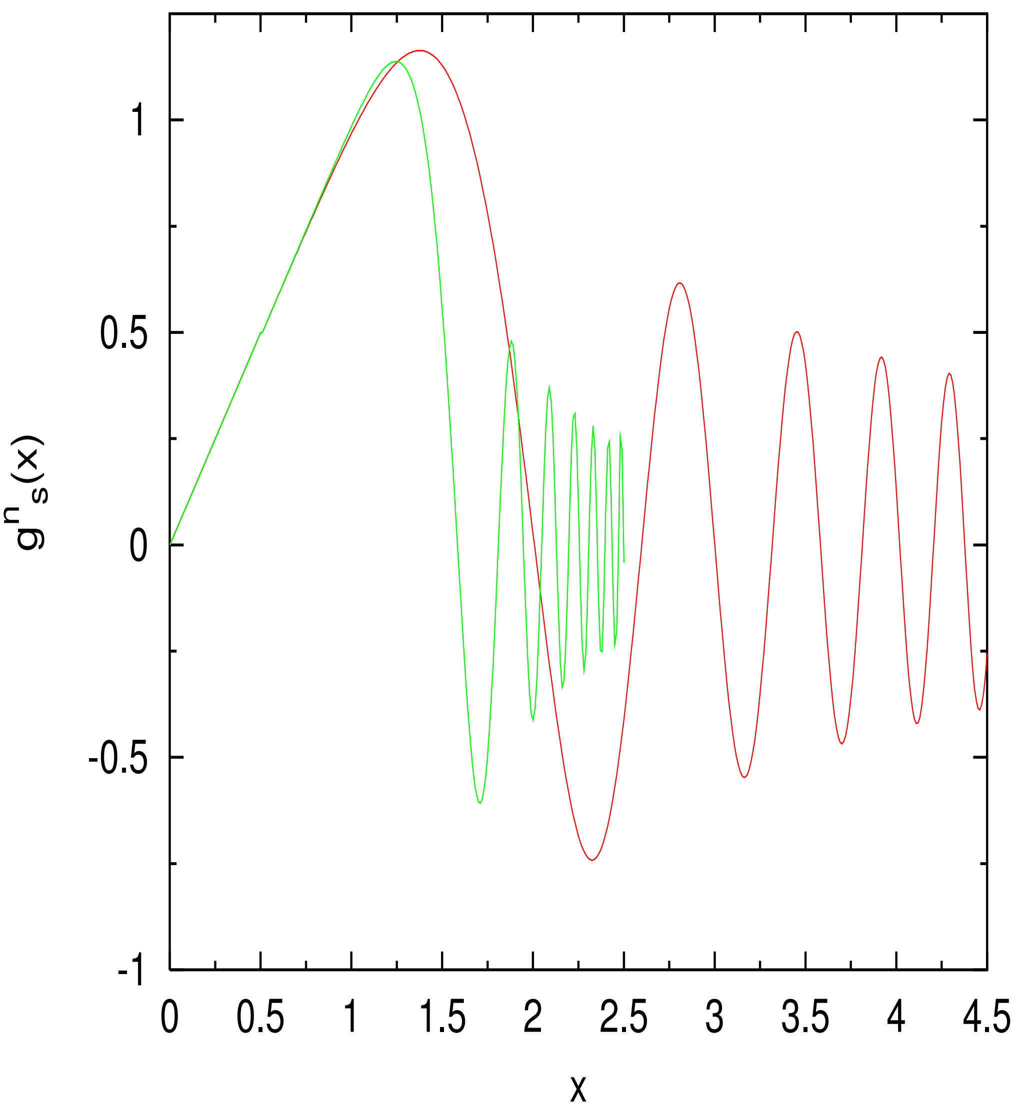


Figure 5: $g_s^n(x)$ vs x for n = -4 (red) and -10 (green). Note that for small x, $g_s^n(x) \sim x$ as expected from equation (14). For larger x, the function oscillates, more for n = -10. The abrupt termination is due to numerical instability.

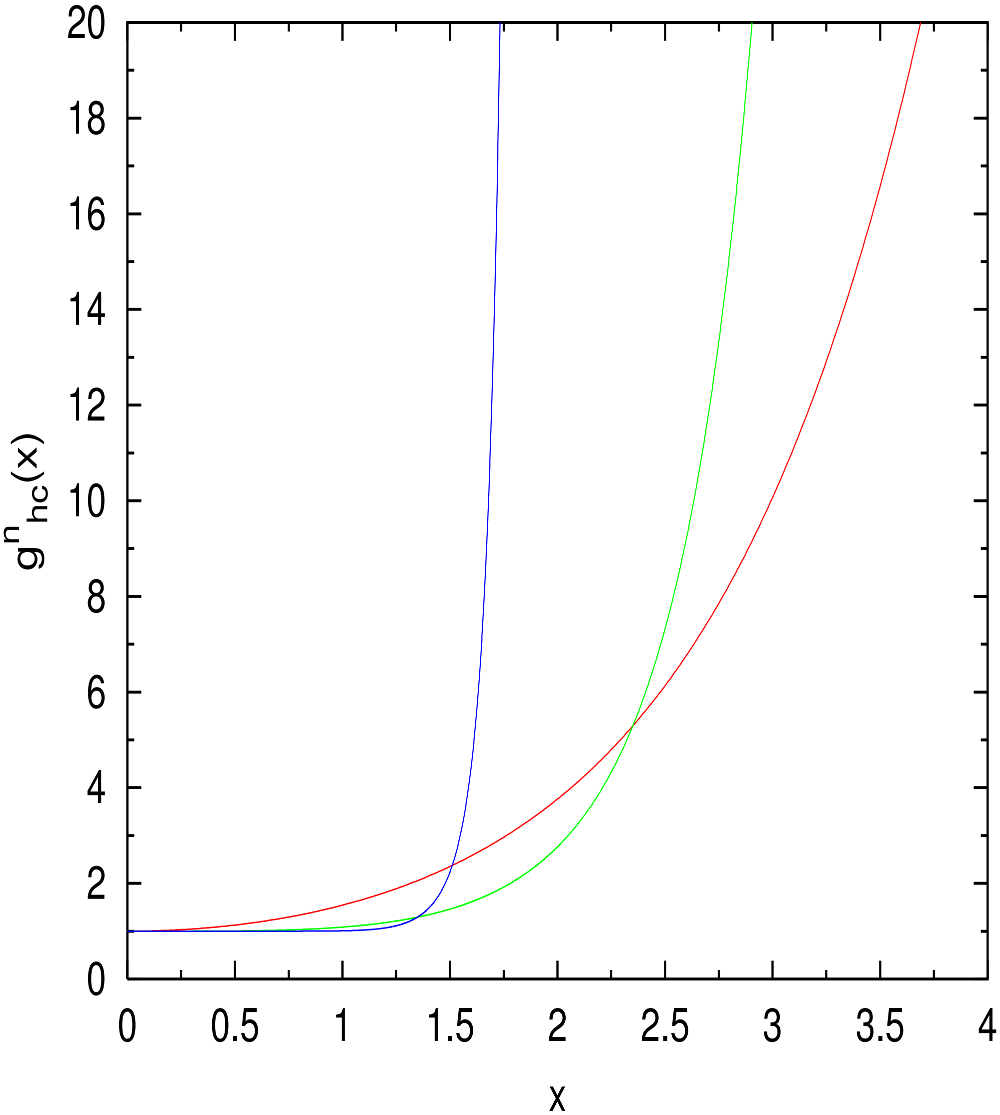


Figure 6: $g^n_{hc}(x)$ vs x for n = 0 (red), 2 (green) and 10 (blue).

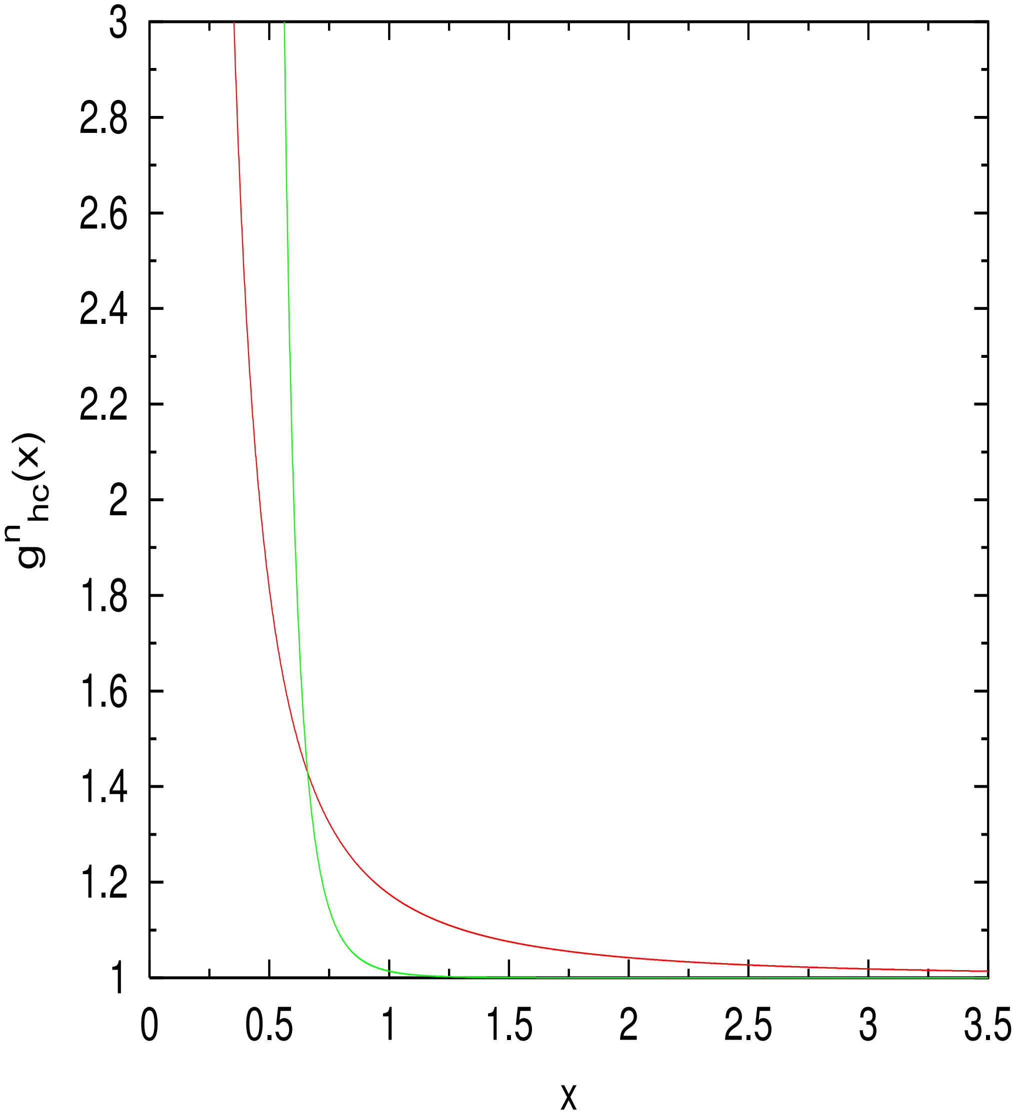


Figure 7: $g^n_{hc}(x)$ vs x for n = -4 (red) and -10 (green). Note the instability near x = 0 and smooth approach towards unity (first term in the series) for large x, as expected.

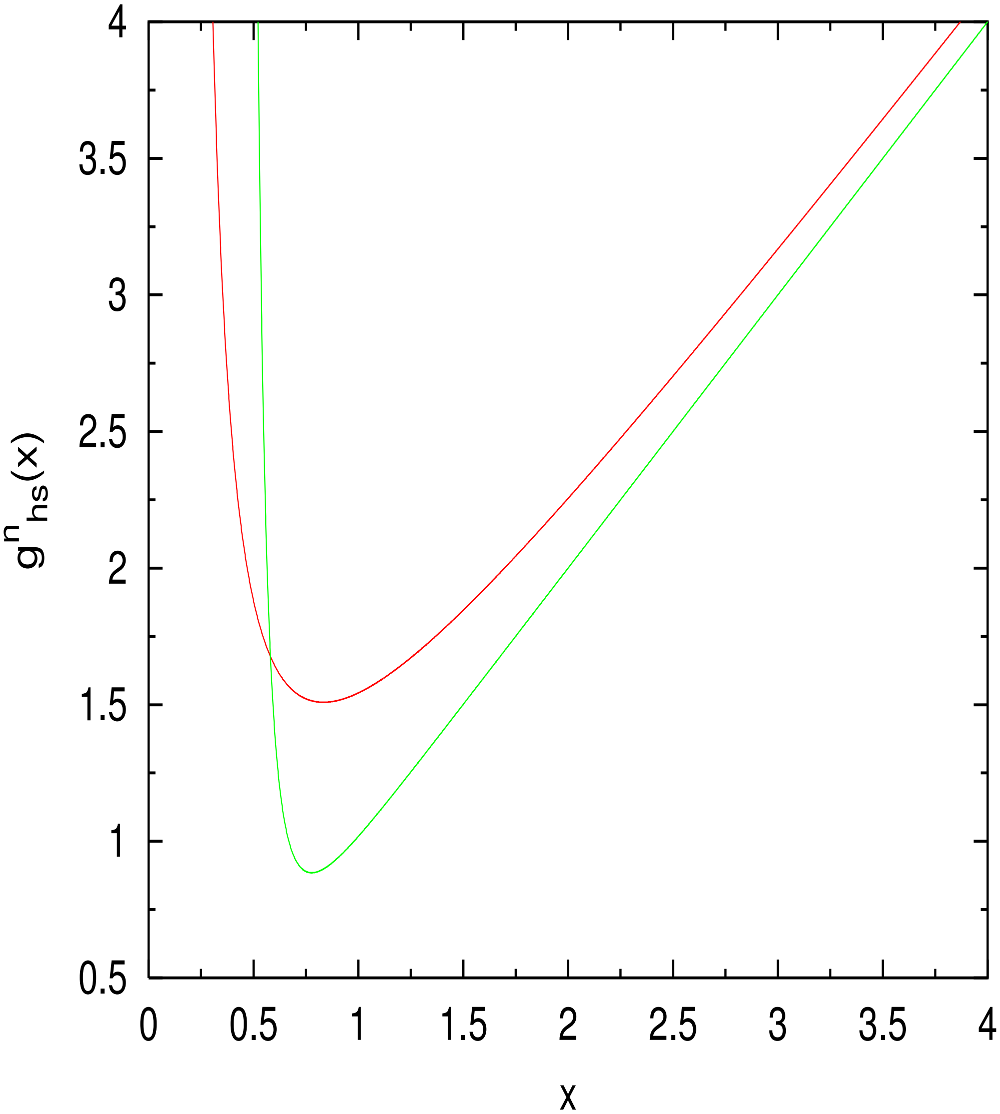


Figure 8: $g^n_{hs}(x)$ vs x for n = -4 (red) and -10 (green). Note the instability near x = 0. For large x, $g^n_{hs}(x) \sim x$ as expected (see also Figure 5).